

On the Stability of the Index of Unbounded Nonlocal Operators in Sobolev Spaces

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Abstract

Unbounded operators corresponding to nonlocal elliptic problems on a bounded region $G \subset \mathbb{R}^2$ are considered. The domain of these operators consists of functions from the Sobolev space $W_2^m(G)$ being generalized solutions of the corresponding $2m$ -order elliptic equation with right-hand side from $L_2(G)$ and satisfying homogeneous nonlocal boundary conditions. It is known that such unbounded operators have the Fredholm property. It is proved in the paper that low-order terms in the differential equation do not affect the index of the operator. Conditions under which nonlocal perturbations on the boundary do not change the index are also formulated.

Introduction

In the one-dimensional case, nonlocal problems were studied by A. Sommerfeld [21], J. D. Tamarkin [22], M. Picone [15]. T. Carleman [2] considered the problem of finding a function harmonic on a two-dimensional bounded domain and subjected to a nonlocal condition connecting the values of this function at different points of the boundary. A. V. Bitsadze and A. A. Smarskii [1] suggested another setting of a nonlocal problem arising in plasma theory: to find a function harmonic on a bounded domain and satisfying nonlocal conditions on shifts of the boundary that can take points of the boundary inside the domain. Different generalizations of the above nonlocal problems were investigated by many authors (see [20] and references therein).

It turns out that the most difficult situation occurs if the support of nonlocal terms intersects the boundary. In this case, solutions of nonlocal problems can have power-law singularities near some points even if the boundary and the right-hand sides are infinitely smooth [16]. For this reason, such problems are naturally studied in weighted spaces (introduced by V. A. Kondrat'ev for boundary-value problems in nonsmooth domains [11]). The most complete theory of nonlocal problems in weighted spaces is developed by A. L. Skubachevskii [16, 17, 18, 19, 20] and his pupils.

Note that the study of nonlocal problems is motivated both by significant theoretical progress in that direction and important applications arising in biophysics, theory of diffusion processes, plasma theory, and so on.

In this paper, we investigate the influence of low-order terms in the elliptic equation and the influence of nonlocal perturbations in boundary conditions upon the index of the unbounded nonlocal operator in $L_2(G)$. This issue was earlier studied by A. L. Skubachevskii [19] for bounded operators in weighted spaces. It is proved in [19] that nonlocal perturbations supported outside the points of conjugation of boundary conditions do not change the index of the corresponding bounded operator. The similar assertion has later been established in Sobolev spaces in the two-dimensional case [5]. In both cases, one can either use the method of continuation with respect to parameter or reduce the

^{*}Supported by the Russian Foundation for Basic Research (project No. 04-01-00256).

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original problem to that where nonlocal perturbations have compact square. As for low-order terms in the elliptic equation, they are simply compact perturbations.

The situation is quite different in the case of unbounded operators. The difficulty is that the low-order terms in elliptic equations are not compact or relatively compact (see Definition A.2); moreover, if the order of the elliptic equation is greater than two, they are not even relatively bounded, and, therefore, they change the domain of definition of the operator. As for nonlocal perturbations in boundary conditions, they explicitly change the domain of definition, and, therefore, they cannot be regarded as compact perturbations (in any sense) either.

To overcome the above difficulties, we consider an auxiliary operator (whose index equals the index of the original operator) acting on weighted spaces. In Sec. 2, we prove that low-order terms in elliptic equations are relatively compact perturbations of the auxiliary operator, and, therefore, they do not affect the index. In Sec. 3, we consider nonlocal perturbations in boundary conditions, which explicitly change the domain of definition. We make use of the notion of a *gap between unbounded operators* (see Definition A.3). We show that, if nonlocal perturbations in boundary conditions satisfy some regularity conditions at the conjugation points, then multiplying the perturbations by a small parameter leads to a small gap between the corresponding operators. Combining this fact with the method of continuation with respect to parameter, we prove the index stability theorem.

Finally, we note that the Fredholm property of unbounded nonlocal operators on $L_2(G)$ was earlier studied either for the case in which nonlocal conditions were set on shifts of the boundary [20] or in the case of a nonlocal perturbation of the Dirichlet problem for a second-order elliptic equation [9, 8]. Elliptic equations of order $2m$ with general nonlocal conditions are being investigated for the first time.

1 Setting of Nonlocal Problems in Bounded Domains

1.1 Setting of nonlocal problems

Let $G \subset \mathbb{R}^2$ be a bounded domain with boundary ∂G . We introduce a set $\mathcal{K} \subset \partial G$ consisting of finitely many points and assume that $\partial G \setminus \mathcal{K} = \bigcup_{i=1}^N \Gamma_i$, where Γ_i are open (in the topology of ∂G) C^∞ -curves. In a neighborhood of each point $g \in \mathcal{K}$, the domain G is supposed to coincide with some plane angle.

For any domain Q and for integer $k \geq 0$, we denote by $W^k(Q) = W_2^k(Q)$ the Sobolev space with the norm

$$\|u\|_{W^k(Q)} = \left(\sum_{|\alpha| \leq k} \int_Q |D^\alpha u|^2 dy \right)^{1/2}$$

(we set $W^0(Q) = L_2(Q)$ for $k = 0$). For an integer $k \geq 1$, we introduce the space $W^{k-1/2}(\Gamma)$ of traces on a smooth curve $\Gamma \subset \overline{Q}$, with the norm

$$\|\psi\|_{W^{k-1/2}(\Gamma)} = \inf \|u\|_{W^k(Q)} \quad (u \in W^k(Q) : u|_\Gamma = \psi). \quad (1.1)$$

For any set $X \in \mathbb{R}^2$ having a nonempty interior, we denote by $C_0^\infty(X)$ the set of functions infinitely differentiable on \overline{X} and compactly supported on X .

Now we introduce different weighted spaces for different domains Q . Consider the following cases: 1. $Q = G$; denote $\mathcal{M} = \mathcal{K}$; 2. Q is a plane angle $K = \{y \in \mathbb{R}^2 : |\omega| < \omega_0\}$, where $0 < \omega_0 < \pi$; denote $\mathcal{M} = \{0\}$; 3. $Q = \{y \in \mathbb{R}^2 : |\omega| < \omega_0, 0 < r < \varepsilon\}$ for some $\varepsilon > 0$; denote $\mathcal{M} = \{0\}$. Introduce the weighted Kondrat'ev space $H_a^k(Q) = H_a^k(Q, \mathcal{M})$ as the completion of the set $C_0^\infty(\overline{Q} \setminus \mathcal{M})$ with

respect to the norm

$$\|u\|_{H_a^k(Q)} = \left(\sum_{|\alpha| \leq k} \int_Q \rho^{2(a+|\alpha|-k)} |D^\alpha u|^2 dx \right)^{1/2},$$

where $k \geq 0$, $a \in \mathbb{R}$, and $\rho(y) = \text{dist}(y, \mathcal{M})$; clearly, $\rho(y) = r$ in Cases 2 and 3 (r being the polar radius).

Denote by $H_a^{k-1/2}(Q)$ ($k \geq 1$ is an integer) the space of traces on a smooth curve $\Gamma \subset \overline{Q}$, with the norm

$$\|\psi\|_{H_a^{k-1/2}(\Gamma)} = \inf \|v\|_{H_a^k(Q)} \quad (v \in H_a^k(Q) : v|_\Gamma = \psi).$$

We denote by $\mathbf{A}(y, D_y)$ and $B_{i\mu s}(y, D_y)$ differential operators of order $2m$ and $m_{i\mu}$ ($m_{i\mu} \leq m-1$), respectively, with complex-valued C^∞ -coefficients ($i = 1, \dots, N$; $\mu = 1, \dots, m$; $s = 0, \dots, S_i$). In particular, we set $\mathbf{B}_{i\mu}^0 u = B_{i\mu 0}(y, D_y)u|_{\Gamma_i}$.

Condition 1.1 (cf., e.g., [13]). *The operator $\mathbf{A}(y, D_y)$ is properly elliptic for all $y \in \overline{G}$, and the system of operators $\{\mathbf{B}_{i\mu}^0\}_{\mu=1}^m$ covers $\mathbf{A}(y, D_y)$ for all $i = 1, \dots, N$ and $y \in \overline{\Gamma_i}$.*

The operators $\mathbf{A}(y, D_y)$ and $\mathbf{B}_{i\mu}^0$ will correspond to a “local” boundary-value problem.

Now we define operators corresponding to nonlocal conditions near the set \mathcal{K} . For $\varepsilon > 0$ and any closed set \mathcal{N} , denote by $\mathcal{O}_\varepsilon(\mathcal{N}) = \{y \in \mathbb{R}^2 : \text{dist}(y, \mathcal{N}) < \varepsilon\}$ its ε -neighborhood.

Let Ω_{is} ($i = 1, \dots, N$; $s = 1, \dots, S_i$) be C^∞ -diffeomorphisms taking some neighborhood \mathcal{O}_i of the curve $\overline{\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})}$ to the set $\Omega_{is}(\mathcal{O}_i)$ in such a way that $\Omega_{is}(\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})) \subset G$ and $\Omega_{is}(g) \in \mathcal{K}$ for $g \in \overline{\Gamma_i} \cap \mathcal{K}$. Thus, under the transformations Ω_{is} , the curves $\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})$ are mapped strictly inside the domain G , whereas the set of end points $\overline{\Gamma_i} \cap \mathcal{K}$ is mapped to itself.

Let us specify the structure of the transformations Ω_{is} near the set \mathcal{K} . Denote by the symbol Ω_{is}^{+1} the transformation $\Omega_{is} : \mathcal{O}_i \rightarrow \Omega_{is}(\mathcal{O}_i)$ and by Ω_{is}^{-1} the inverse transformation. The set of all points $\Omega_{i_q s_q}^{\pm 1}(\dots \Omega_{i_1 s_1}^{\pm 1}(g)) \in \mathcal{K}$ ($1 \leq s_j \leq S_{i_j}$, $j = 1, \dots, q$), i.e., the set of all points that can be obtained by consecutively applying the transformations $\Omega_{i_j s_j}^{+1}$ or $\Omega_{i_j s_j}^{-1}$ (taking the points of \mathcal{K} to \mathcal{K}) to the point $g \in \mathcal{K}$, is called an *orbit* of the point g and is denoted by $\text{Orb}(g)$.

Clearly, for any $g, g' \in \mathcal{K}$ either $\text{Orb}(g) = \text{Orb}(g')$ or $\text{Orb}(g) \cap \text{Orb}(g') = \emptyset$. In what follows, we assume that the set \mathcal{K} consists of one orbit. (All results can be directly generalized to the case in which \mathcal{K} consists of finitely many mutually disjoint orbits, see Remark 3.3.) Denote the points of the set (orbit) \mathcal{K} by g_j , $j = 1, \dots, N$.

Take a small number ε such that there exist neighborhoods $\mathcal{O}_{\varepsilon_1}(g_j)$ of the points $g_j \in \mathcal{K}$ satisfying the following conditions: 1. $\mathcal{O}_{\varepsilon_1}(g_j) \supset \mathcal{O}_\varepsilon(g_j)$; 2. the boundary ∂G coincides with some plane angle in the neighborhood $\mathcal{O}_{\varepsilon_1}(g_j)$; 3. $\overline{\mathcal{O}_{\varepsilon_1}(g_j)} \cap \overline{\mathcal{O}_{\varepsilon_1}(g_k)} = \emptyset$ for any $g_j, g_k \in \mathcal{K}$, $j \neq k$; 4. if $g_j \in \overline{\Gamma_i}$ and $\Omega_{is}(g_j) = g_k$, then $\mathcal{O}_\varepsilon(g_j) \subset \mathcal{O}_i$ and $\Omega_{is}(\mathcal{O}_\varepsilon(g_j)) \subset \mathcal{O}_{\varepsilon_1}(g_k)$.

For each point $g_j \in \overline{\Gamma_i} \cap \mathcal{K}$, we fix a transformation $Y_j : y \mapsto y'(g_j)$ which is the composition of the shift by the vector $-\overrightarrow{Og_j}$ and the rotation through some angle so that

$$Y_j(\mathcal{O}_{\varepsilon_1}(g_j)) = \mathcal{O}_{\varepsilon_1}(0), \quad Y_j(G \cap \mathcal{O}_{\varepsilon_1}(g_j)) = K_j \cap \mathcal{O}_{\varepsilon_1}(0),$$

$$Y_j(\Gamma_i \cap \mathcal{O}_{\varepsilon_1}(g_j)) = \gamma_{j\sigma} \cap \mathcal{O}_{\varepsilon_1}(0) \quad (\sigma = 1 \text{ or } \sigma = 2),$$

where $K_j = \{y \in \mathbb{R}^2 : r > 0, |\omega| < \omega_j\}$, $\gamma_{j\sigma} = \{y \in \mathbb{R}^2 : r > 0, \omega = (-1)^\sigma \omega_j\}$, (ω, r) are the polar coordinates, and $0 < \omega_j < \pi$.

Condition 1.2. *Let $g_j \in \overline{\Gamma_i} \cap \mathcal{K}$ and $\Omega_{is}(g_j) = g_k \in \mathcal{K}$; then the transformation $Y_k \circ \Omega_{is} \circ Y_j^{-1} : \mathcal{O}_\varepsilon(0) \rightarrow \mathcal{O}_{\varepsilon_1}(0)$ is the composition of a rotation and a homothety.*

Remark 1.1. In particular, Condition 1.2, being combined with the assumption $\Omega_{is}(\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})) \subset G$, means that, if $g \in \Omega_{is}(\overline{\Gamma_i} \cap \mathcal{K}) \cap \overline{\Gamma_j} \cap \mathcal{K} \neq \emptyset$, then the curves $\Omega_{is}(\overline{\Gamma_i})$ and $\overline{\Gamma_j}$ are not tangent to each other at the point g .

Consider a number ε_0 , $0 < \varepsilon_0 \leq \varepsilon$, satisfying the following condition: if $g_j \in \overline{\Gamma_i}$ and $\Omega_{is}(g_j) = g_k$, then $\mathcal{O}_{\varepsilon_0}(g_k) \subset \Omega_{is}(\mathcal{O}_\varepsilon(g_j))$. Introduce a function $\zeta \in C^\infty(\mathbb{R}^2)$ such that $\zeta(y) = 1$ for $y \in \mathcal{O}_{\varepsilon_0/2}(\mathcal{K})$ and $\text{supp } \zeta \subset \mathcal{O}_{\varepsilon_0}(\mathcal{K})$.

Now we define nonlocal operators $\mathbf{B}_{i\mu}^1$ by the formula

$$\mathbf{B}_{i\mu}^1 u = \sum_{s=1}^{S_i} (B_{i\mu s}(y, D_y)(\zeta u))(\Omega_{is}(y)), \quad y \in \Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K}), \quad \mathbf{B}_{i\mu}^1 u = 0, \quad y \in \Gamma_i \setminus \mathcal{O}_\varepsilon(\mathcal{K}),$$

where $(B_{i\mu s}(y, D_y)u)(\Omega_{is}(y)) = B_{i\mu s}(x, D_x)u(x)|_{x=\Omega_{is}(y)}$. Since $\mathbf{B}_{i\mu}^1 u = 0$ whenever $\text{supp } u \subset \overline{G} \setminus \overline{\mathcal{O}_{\varepsilon_0}(\mathcal{K})}$, we say that the operators $\mathbf{B}_{i\mu}^1$ correspond to nonlocal terms supported near the set \mathcal{K} .

For any $\rho > 0$, we denote $G_\rho = \{y \in G : \text{dist}(y, \partial G) > \rho\}$. Consider operators $\mathbf{B}_{i\mu}^2$ satisfying the following condition (cf. [16, 19, 4]).

Condition 1.3. *There exist numbers $\varkappa_1 > \varkappa_2 > 0$ and $\rho > 0$ such that the following inequalities hold:*

$$\|\mathbf{B}_{i\mu}^2 u\|_{W^{2m-m_{i\mu}-1/2}(\Gamma_i)} \leq c_1 \|u\|_{W^{2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})}, \quad (1.2)$$

$$\|\mathbf{B}_{i\mu}^2 u\|_{W^{2m-m_{i\mu}-1/2}(\Gamma_i \setminus \overline{\mathcal{O}_{\varkappa_2}(\mathcal{K})})} \leq c_2 \|u\|_{W^{2m}(G_\rho)}. \quad (1.3)$$

Remark 1.2. In (1.2), (1.3), and throughout the paper, we denote by c, c_1, c_2, \dots and k_1, k_2, \dots positive constants which do not depend on the functions entering the corresponding inequality.

We assume that Conditions 1.1–1.3 hold throughout, including the formulation of lemmas.

It follows from (1.2) that $\mathbf{B}_{i\mu}^2 u = 0$ whenever $\text{supp } u \subset \mathcal{O}_{\varkappa_1}(\mathcal{K})$. For this reason, we say that the operators $\mathbf{B}_{i\mu}^2$ correspond to nonlocal terms supported outside the set \mathcal{K} .

We study the following nonlocal elliptic problem:

$$\mathbf{A}(y, D_y)u = f(y) \quad (y \in G), \quad (1.4)$$

$$\mathbf{B}_{i\mu} u \equiv \mathbf{B}_{i\mu}^0 u + \mathbf{B}_{i\mu}^1 u + \mathbf{B}_{i\mu}^2 u = 0 \quad (y \in \Gamma_i; \quad i = 1, \dots, N; \quad \mu = 1, \dots, m), \quad (1.5)$$

where $f \in L_2(G)$. Introduce the space $W^m(G, \mathbf{B})$ consisting of functions $u \in W^m(G)$ that satisfy homogeneous nonlocal conditions (1.5). Consider the unbounded operator $\mathbf{P} : D(\mathbf{P}) \subset L_2(G) \rightarrow L_2(G)$ given by

$$\mathbf{P}u = \mathbf{A}(y, D_y)u, \quad u \in D(\mathbf{P}) = \{u \in W^m(G, \mathbf{B}) : \mathbf{A}(y, D_y)u \in L_2(G)\}.$$

Definition 1.1. A function u is called a *generalized solution* of problem (1.4), (1.5) with right-hand side $f \in L_2(G)$ if $u \in D(\mathbf{P})$ and $\mathbf{P}u = f$.

Equivalent definition of a generalized solution can be given in terms of an integral identity [7].

Note that generalized solutions a priori belong to the space $W^m(G)$, whereas Condition 1.3 is formulated for functions belonging to the space W^{2m} outside the set \mathcal{K} . Such a formulation can be justified by the following result (see Lemma 2.1 in [7] and Lemma 5.1 in [5]).

Lemma 1.1. *Let $u \in W^m(G)$ be a generalized solution of problem (1.4), (1.5) with right-hand side $f \in W^k(G)$. Then*

$$\|u\|_{W^{k+2m}(G \setminus \overline{\mathcal{O}_{\delta_1}(\mathcal{K})})} \leq c_\delta \left(\|f\|_{W^k(G \setminus \overline{\mathcal{O}_{\delta_1}(\mathcal{K})})} + \|u\|_{L_2(G)} \right) \quad \forall \delta > 0,$$

where $\delta_1 = \delta_1(\delta) > 0$ and $c_\delta > 0$ do not depend on u .

Theorem 1.1 (see Theorem 2.1 in [7]). *Let Conditions 1.1–1.3 hold. Then the operator \mathbf{P} has the Fredholm property.*¹

The aim of this paper is to investigate the influence of lower-order terms in (1.4) and nonlocal operators $\mathbf{B}_{i\mu}^1$ and $\mathbf{B}_{i\mu}^2$ in (1.5) upon the index of the operator \mathbf{P} .

1.2 Nonlocal Problems near the Set \mathcal{K}

When studying problem (1.4), (1.5), one must pay particular attention to the behavior of solutions near the set \mathcal{K} of conjugation points. Let us consider the corresponding model problems. Denote by $u_j(y)$ the function $u(y)$ for $y \in \mathcal{O}_{\varepsilon_1}(g_j)$. If $g_j \in \overline{\Gamma}_i$, $y \in \mathcal{O}_\varepsilon(g_j)$, and $\Omega_{is}(y) \in \mathcal{O}_{\varepsilon_1}(g_k)$, then we denote the function $u(\Omega_{is}(y))$ by $u_k(\Omega_{is}(y))$. In this notation, nonlocal problem (1.4), (1.5) acquires the following form in the ε -neighborhood of the set (orbit) \mathcal{K} :

$$\begin{aligned} \mathbf{A}(y, D_y)u_j &= f(y) \quad (y \in \mathcal{O}_\varepsilon(g_j) \cap G), \\ B_{i\mu 0}(y, D_y)u_j(y)|_{\mathcal{O}_\varepsilon(g_j) \cap \Gamma_i} &+ \sum_{s=1}^{S_i} (B_{i\mu s}(y, D_y)(\zeta u_k))(\Omega_{is}(y))|_{\mathcal{O}_\varepsilon(g_j) \cap \Gamma_i} = f_{i\mu}(y) \\ (y \in \mathcal{O}_\varepsilon(g_j) \cap \Gamma_i; \quad i \in \{1 \leq i \leq N : g_j \in \overline{\Gamma}_i\}; \quad j = 1, \dots, N; \quad \mu = 1, \dots, m), \end{aligned}$$

where $f_{i\mu} = -\mathbf{B}_{i\mu}^2 u$.

Let $y \mapsto y'(g_j)$ be the change of variables described in Sec. 1.1. Denote $K_j^\varepsilon = K_j \cap \mathcal{O}_\varepsilon(0)$, $\gamma_{j\sigma}^\varepsilon = \gamma_{j\sigma} \cap \mathcal{O}_\varepsilon(0)$. Introduce the functions

$$U_j(y') = u_j(y(y')), \quad f_j(y') = f(y(y')), \quad y' \in K_j^\varepsilon, \quad f_{j\sigma\mu}(y') = f_{i\mu}(y(y')), \quad y' \in \gamma_{j\sigma}^\varepsilon,$$

where $\sigma = 1$ ($\sigma = 2$) if, under the transformation $y \mapsto y'(g_j)$, the curve Γ_i is mapped to the side γ_{j1} (γ_{j2}) of the angle K_j . Denote y' by y again. Then, by virtue of Condition 1.2, problem (1.4), (1.5) acquires the form

$$\mathbf{A}_j(y, D_y)U_j = f_j(y) \quad (y \in K_j^\varepsilon), \tag{1.6}$$

$$\sum_{k,s} (B_{j\sigma\mu ks}(y, D_y)U_k)(\mathcal{G}_{j\sigma ks}y) = f_{j\sigma\mu}(y) \quad (y \in \gamma_{j\sigma}^\varepsilon); \tag{1.7}$$

here $j, k = 1, \dots, N$; $\sigma = 1, 2$; $\mu = 1, \dots, m$; $s = 0, \dots, S_{j\sigma k}$; $\mathbf{A}_j(y, D_y)$ and $B_{j\sigma\mu ks}(y, D_y)$ are differential operators of order $2m$ and $m_{j\sigma\mu}$ ($m_{j\sigma\mu} \leq m - 1$), respectively, with C^∞ complex-valued coefficients; $\mathcal{G}_{j\sigma ks}$ is the operator of rotation by an angle $\omega_{j\sigma ks}$ and the homothety with a coefficient $\chi_{j\sigma ks}$ ($\chi_{j\sigma ks} > 0$). Moreover, $|(-1)^\sigma b_j + \omega_{j\sigma ks}| < b_k$ for $(k, s) \neq (j, 0)$ (cf. Remark 1.1) and $\omega_{j\sigma j0} = 0$, $\chi_{j\sigma j0} = 1$ (i.e., $\mathcal{G}_{j\sigma j0}y \equiv y$).

Set $D_\chi = 2 \max\{\chi_{j\sigma ks}\}$. The following lemma establishes the regularity property for solutions of nonlocal problems near the set \mathcal{K} .

Lemma 1.2 (see² Lemma 2.3 in [7]). *Let (U_1, \dots, U_N) be a solution of problem (1.6), (1.7) such that*

$$U_j \in W^{2m}(K_j^{D_\chi \varepsilon} \cap \{|y| > \delta\}) \quad \forall \delta > 0, \quad U_j \in H_{a-2m}^0(K_j^{D_\chi \varepsilon}),$$

where $a \in \mathbb{R}$. Suppose that $f_j \in H_a^0(K_j^\varepsilon)$ and $f_{j\sigma\mu} \in H_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^\varepsilon)$. Then

$$\sum_j \|U_j\|_{H_a^{2m}(K_j^{\varepsilon/D_\chi^3})} \leq c \sum_j \left(\|f_j\|_{H_a^0(K_j^\varepsilon)} + \sum_{\sigma,\mu} \|f_{j\sigma\mu}\|_{H_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^\varepsilon)} + \|U_j\|_{H_{a-2m}^0(K_j^\varepsilon)} \right).$$

¹See Definition A.1.

²Lemma 2.3 in [7] was formulated for $a > 2m - 1$. However, its proof remains true for any $a \in \mathbb{R}$.

We write the principal homogeneous parts of the operators $\mathbf{A}_j(0, D_y)$ and $B_{j\sigma\mu ks}(0, D_y)$ in the polar coordinates, $r^{-2m}\tilde{\mathcal{A}}_j(\omega, D_\omega, rD_r)$, $r^{-m_{j\sigma\mu}}\tilde{B}_{j\sigma\mu ks}(\omega, D_\omega, rD_r)$, respectively, and consider the analytic operator-valued function

$$\tilde{\mathcal{L}}(\lambda) : \prod_{j=1}^N W^{l+2m}(-\omega_j, \omega_j) \rightarrow \prod_{j=1}^N (W^l(-\omega_j, \omega_j) \times \mathbb{C}^{2m}),$$

$$\tilde{\mathcal{L}}(\lambda)\varphi = \left\{ \tilde{\mathcal{A}}_j(\omega, D_\omega, \lambda)\varphi_j, \sum_{k,s} (\chi_{j\sigma ks})^{i\lambda - m_{j\sigma\mu}} \tilde{B}_{j\sigma\mu ks}(\omega, D_\omega, \lambda)\varphi_k(\omega + \omega_{j\sigma ks})|_{\omega=(-1)^\sigma\omega_j} \right\}.$$

Basic definitions and facts concerning eigenvalues, eigenvectors, and associate vectors of analytic operator-valued functions can be found in [3]. In the sequel, it will be on principle that the spectrum of the operator $\tilde{\mathcal{L}}(\lambda)$ is discrete (see Lemma 2.1 in [17]).

2 Perturbations by Lower-Order Terms

2.1 Reduction to weighted spaces

Introduce the lower-order terms operator

$$A'(y, D_y) = \sum_{|\alpha| \leq 2m-1} a_\alpha(y) D^\alpha, \quad (2.1)$$

where $a_\alpha \in C^\infty(\mathbb{R}^2)$. Consider the perturbed operator $\mathbf{P}' : D(\mathbf{P}') \subset L_2(G) \rightarrow L_2(G)$ given by

$$\mathbf{P}'u = \mathbf{A}(y, D_y)u + A'(y, D_y)u, \quad u \in D(\mathbf{P}') = \{u \in W^m(G, \mathbf{B}) : \mathbf{A}(y, D_y)u + A'(y, D_y)u \in L_2(G)\}.$$

By Theorem 1.1, the unbounded operator \mathbf{P}' has the Fredholm property (just as \mathbf{P} has). The main result of this section (to be proved in Sec. 2.2) is as follows.

Theorem 2.1. *Let Conditions 1.1–1.3 hold. Then $\text{ind } \mathbf{P}' = \text{ind } \mathbf{P}$.*

This theorem shows that the lower-order terms in (1.4) do not affect the index of the unbounded operator \mathbf{P} . The difficulty is that the above perturbations are, in general, neither compact nor \mathbf{P} -compact in the sense of Definition A.2. If $m = 1$ then $u \in D(\mathbf{P})$ implies only $u \in W^1(G)$, which ensures the \mathbf{P} -boundedness of the perturbation but not its \mathbf{P} -compactness. However, if $m \geq 2$, then $u \in D(\mathbf{P})$ does not imply $u \in W^{2m-1}(G)$, and the perturbation is not even \mathbf{P} -bounded. Moreover, $D(\mathbf{P}') \neq D(\mathbf{P})$ in the latter case.

To overcome this difficulty, we introduce the operator $\mathbf{Q} : D(\mathbf{Q}) \subset L_2(G) \rightarrow H_a^0(G)$ given by

$$\mathbf{Q}u = \mathbf{A}(y, D_y)u, \quad u \in D(\mathbf{Q}) = \{u \in W^m(G, \mathbf{B}) : \mathbf{A}(y, D_y)u \in H_a^0(G)\}. \quad (2.2)$$

In this definition and further (unless otherwise stated), we assume that

$$m - 1 < a < m.$$

We will prove that $\text{ind } \mathbf{Q} = \text{ind } \mathbf{P}$. On the other hand, we will show that the operator $A'(y, D_y)$ is a \mathbf{Q} -compact perturbation, and, therefore, it does not change the index of \mathbf{Q} and hence \mathbf{P} .

Lemma 2.1. *Let the line $\text{Im } \lambda = a + 1 - 2m$ contain no eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. Then the operator \mathbf{Q} has the Fredholm property and $\text{ind } \mathbf{Q} = \text{ind } \mathbf{P}$.*

Proof. 1. It is shown in [5, Sec. 6] that $\mathbf{B}_{i\mu}u \in H_a^{2m-m_{i\mu}-1/2}(\Gamma_i) \dot{+} R_a^{i\mu}(\Gamma_i)$ for $u \in H_a^{2m}(G)$, where $R_a^{i\mu}(\Gamma_i)$ is a finite-dimensional subspace in $H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)$ for any $a' > 2m - 1$. Set

$$\mathcal{H}_a^0(G, \Gamma) = H_a^0(G) \times \prod_{i=1}^N \prod_{\mu=1}^m H_a^{2m-m_{i\mu}-1/2}(\Gamma_i), \quad \mathcal{R}_a^0(G, \Gamma) = \{0\} \times \prod_{i=1}^N \prod_{\mu=1}^m R_a^{i\mu}(\Gamma_i).$$

By Theorem 6.1 in [5], the bounded operator

$$\mathbf{L} = \{\mathbf{A}(y, D_y), \mathbf{B}_{i\mu}\} : H_a^{2m}(G) \rightarrow \mathcal{H}_a^0(G, \Gamma) \dot{+} \mathcal{R}_a^0(G, \Gamma) \quad (2.3)$$

has the Fredholm property. Therefore, by virtue of the compactness of the embedding $H_a^{2m}(G) \subset L_2(G)$ (see Lemma A.1) and by Theorem A.1, we have

$$\|u\|_{H_a^{2m}(G)} \leq k_1 (\|\mathbf{L}u\|_{\mathcal{H}_a^0(G, \Gamma) \dot{+} \mathcal{R}_a^0(G, \Gamma)} + \|u\|_{L_2(G)}). \quad (2.4)$$

2. Introduce the unbounded operator $\dot{\mathbf{Q}} : D(\dot{\mathbf{Q}}) \subset L_2(G) \rightarrow H_a^0(G)$ given by

$$\dot{\mathbf{Q}}u = \mathbf{A}(y, D_y)u, \quad u \in D(\dot{\mathbf{Q}}) = \{u \in H_a^{2m}(G) : \mathbf{B}_{i\mu}u = 0\}. \quad (2.5)$$

Since $H_a^{2m}(G) \subset W^m(G)$, it follows that $\dot{\mathbf{Q}}$ is a restriction of \mathbf{Q} , i.e., $\dot{\mathbf{Q}} \subset \mathbf{Q}$.

First, we prove that $\dot{\mathbf{Q}}$ has the Fredholm property. Let $u \in D(\dot{\mathbf{Q}})$; then $u \in D(\mathbf{L}) = H_a^{2m}(G)$ and $\mathbf{A}(y, D_y)u \in H_a^0(G)$, $\mathbf{B}_{i\mu}u = 0$. Therefore, estimate (2.4) acquires the form

$$\|u\|_{H_a^{2m}(G)} \leq k_1 (\|\dot{\mathbf{Q}}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}) \quad \forall u \in D(\dot{\mathbf{Q}}). \quad (2.6)$$

It follows from (2.6) that the operator $\dot{\mathbf{Q}}$ is closed, $\dim \ker \dot{\mathbf{Q}} < \infty$, and $\mathcal{R}(\dot{\mathbf{Q}}) = \overline{\mathcal{R}(\dot{\mathbf{Q}})}$ (to obtain the latter two properties, one must apply Theorem A.1).

Let us prove that $\text{codim } \mathcal{R}(\dot{\mathbf{Q}}) < \infty$. Since \mathbf{L} has the Fredholm property, there exist finitely many linearly independent functions $F_1, \dots, F_d \in H_a^0(G)$ such that a function $f \in H_a^0(G)$ belongs to the image of $\dot{\mathbf{Q}}$ iff $(f, F_j)_{H_a^0(G)} = 0$, $j = 1, \dots, d$. Thus, $\dot{\mathbf{Q}}$ has the Fredholm property.

3. Now we prove that \mathbf{Q} has the Fredholm property. Since $\ker \mathbf{Q} = \ker \mathbf{P}$ and \mathbf{P} has the Fredholm property, it follows that

$$\dim \ker \mathbf{Q} = \dim \ker \mathbf{P} < \infty. \quad (2.7)$$

On the other hand, \mathbf{Q} is an extension of the Fredholm operator $\dot{\mathbf{Q}}$; therefore,

$$\mathcal{R}(\mathbf{Q}) = \overline{\mathcal{R}(\dot{\mathbf{Q}})}, \quad \text{codim } \mathcal{R}(\mathbf{Q}) < \infty. \quad (2.8)$$

Thus, \mathbf{Q} is an extension of the Fredholm operator $\dot{\mathbf{Q}}$, and possesses properties (2.7) and (2.8). Applying Theorem A.2, we see that \mathbf{Q} has the Fredholm property.

4. By virtue of (2.7), it remains to prove that $\text{codim } \mathcal{R}(\mathbf{Q}) = \text{codim } \mathcal{R}(\mathbf{P})$.

Let $\text{codim } \mathcal{R}(\mathbf{Q}) = d_1$, where $d_1 \leq d$. Take an arbitrary function $f \in L_2(G)$. Then $f \in \mathcal{R}(\mathbf{P})$ iff $f \in \mathcal{R}(\mathbf{Q})$ because $L_2(G) \subset H_a^0(G)$. However, the belonging $f \in \mathcal{R}(\mathbf{Q})$ is equivalent to the relations $(f, F_j)_{H_a^0(G)} = 0$, $j = 1, \dots, d_1$, where $F_1, \dots, F_{d_1} \in H_a^0(G)$ are linearly independent functions. Using Schwarz' inequality, the boundedness of the embedding $L_2(G) \subset H_a^0(G)$, and Riesz' theorem, we see that these relations are equivalent to the following ones: $(f, f_j)_{L_2(G)} = 0$, $j = 1, \dots, d_1$, where $f_j \in L_2(G)$. Moreover, the functions f_1, \dots, f_{d_1} are linearly independent. (Otherwise, some linear combination of the functions F_1, \dots, F_{d_1} would be orthogonal in $H_a^0(G)$ to any function from $L_2(G)$. This is impossible because F_1, \dots, F_{d_1} are linearly independent, while $L_2(G)$ is dense in $H_a^0(G)$.) Thus, we have proved that $\text{codim } \mathcal{R}(\mathbf{P}) = d_1$. \square

Introduce the perturbed operator $\mathbf{Q}' : D(\mathbf{Q}') \subset L_2(G) \rightarrow H_a^0(G)$ given by

$$\mathbf{Q}'u = \mathbf{A}(y, D_y)u + A'(y, D_y)u, \quad u \in D(\mathbf{Q}') = \{u \in W^m(G, \mathbf{B}) : \mathbf{A}(y, D_y)u + A'(y, D_y)u \in H_a^0(G)\}.$$

In the following section, we prove that $\text{ind } \mathbf{Q}' = \text{ind } \mathbf{Q}$, provided that the line $\text{Im } \lambda = a + 1 - 2m$ contains no eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. Then, using the discreteness of the spectrum of $\tilde{\mathcal{L}}(\lambda)$ and Lemma 2.1, we will prove Theorem 2.1.

2.2 Compactness of lower-order terms in weighted spaces

Lemma 2.2. *Let the line $\text{Im } \lambda = a + 1 - 2m$ contain no eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. Then*

$$\|u\|_{W^m(G)} \leq c (\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}) \quad \forall u \in D(\mathbf{Q}).$$

Proof. Consider the unbounded operator $\hat{\mathbf{Q}} : D(\hat{\mathbf{Q}}) \subset W^m(G) \rightarrow H_a^0(G)$ given by $\hat{\mathbf{Q}}u = \mathbf{A}(y, D_y)u$, $u \in D(\hat{\mathbf{Q}}) = D(\mathbf{Q})$. Since \mathbf{Q} has the Fredholm property, the same is true for $\hat{\mathbf{Q}}$. Therefore, the desired estimate follows from the compactness of the embedding $W^m(G) \subset L_2(G)$ and from Theorem A.1. \square

Take a number b such that

$$m - 1 < b < a < m. \quad (2.9)$$

Consider a function $\psi_j \in C_0^\infty(\mathbb{R}^2)$ equal to 1 in a small neighborhood of the point $g_j \in \mathcal{K}$ and vanishing outside a larger neighborhood of g_j . The following lemma describes the behavior of $u \in D(\mathbf{Q})$ near the set \mathcal{K} .

Lemma 2.3. *For any $u \in D(\mathbf{Q})$, we have*

$$u(y) = \sum_{j=1}^N P_j(y) + v(y), \quad (2.10)$$

where

$$P_j(y) = \psi_j(y) \sum_{|\alpha| \leq m-2} p_{j\alpha} (y - g_j)^\alpha, \quad p_{j\alpha} \in \mathbb{C}, \quad (2.11)$$

and $v \in H_{b+1}^{2m}(G)$ (if $m = 1$, we set $P_j(y) \equiv 0$); moreover,

$$\sum_{j,\alpha} |p_{j\alpha}| + \|v\|_{H_{b+1}^{2m}(G)} \leq c (\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}). \quad (2.12)$$

Proof. 1. It follows from Lemma 1.1 that $u \in W^{2m}(G \setminus \overline{\mathcal{O}_\delta(\mathcal{K})})$ for any $\delta > 0$ and

$$\|u\|_{W^{2m}(G \setminus \overline{\mathcal{O}_\delta(\mathcal{K})})} \leq k_{1\delta} (\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}), \quad (2.13)$$

where $k_{1\delta}$ does not depend on u . Therefore, it suffices to consider the behavior of u near the set \mathcal{K} .

By Lemma A.2, $u \in W^m(G)$ can be represented in the form (2.10), where $P_j(y)$ is given by (2.11), $v \in H_{b-m+1}^m(G)$, and

$$\sum_{j,\alpha} |p_{j\alpha}| + \|v\|_{H_{b-m+1}^m(G)} \leq k_2 (\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}) \quad (2.14)$$

(to obtain (2.14), we have also applied Lemma 2.2).

Moreover, relations (2.10), (2.13), and (2.14) imply that

$$\|v\|_{W^{2m}(G \setminus \overline{\mathcal{O}_\delta(\mathcal{K})})} \leq k_{2\delta} (\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}) \quad \forall \delta > 0, \quad (2.15)$$

where $k_{2\delta}$ does not depend on u . It remains to prove that $v \in H_{b+1}^{2m}(G)$.

2. By using (1.4) and (1.5), we see that v is a solution of the problem

$$\mathbf{A}(y, D_y)v = f - \mathbf{A}(y, D_y)P \equiv f', \quad \mathbf{B}_{i\mu}^0 v + \mathbf{B}_{i\mu}^1 v = -\mathbf{B}_{i\mu} P - \mathbf{B}_{i\mu}^2 v \equiv f'_{i\mu}, \quad (2.16)$$

where $P(y) = \sum_{j=1}^N P_j(y)$ and $f = \mathbf{Q}u \in H_a^0(G)$. It follows from the boundedness of the embedding $H_a^0(G) \subset H_{b+1}^0(G)$ (see (2.9)) and from the estimate of the coefficients $p_{j\alpha}$ (see (2.14)) that

$$\|f'\|_{H_{b+1}^0(G)} \leq k_3 (\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}). \quad (2.17)$$

Similarly, using additionally inequalities (1.2) and (2.15), we obtain $f'_{i\mu} = -\mathbf{B}_{i\mu} P - \mathbf{B}_{i\mu}^2 v \in W^{2m-m_{i\mu}-1/2}(\Gamma_i)$ and

$$\|f'_{i\mu}\|_{W^{2m-m_{i\mu}-1/2}(\Gamma_i)} \leq k_4 (\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}). \quad (2.18)$$

On the other hand, $v \in H_{b-m+1}^m(G)$; hence $f'_{i\mu} = \mathbf{B}_{i\mu}^0 v + \mathbf{B}_{i\mu}^1 v \in H_{b-m+1}^{m-m_{i\mu}-1/2}(\Gamma_i)$. We claim that

$$\|f'_{i\mu}\|_{H_{b+1}^{2m-m_{i\mu}-1/2}(\Gamma_i)} \leq k_5 (\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}). \quad (2.19)$$

To prove this assertion, we fix i and μ and set $\Gamma = \Gamma_i$. Let $g \in \bar{\Gamma} \setminus \Gamma$. Assume, without loss of generality, that $g = 0$ and Γ coincides with the axis Oy_1 in a sufficiently small neighborhood $\mathcal{O}_\varepsilon(0)$ of the origin. Denote

$$G^\varepsilon = G \cap \mathcal{O}_\varepsilon(0), \quad \Gamma^\varepsilon = \Gamma \cap \mathcal{O}_\varepsilon(0),$$

in which case $H_a^k(G^\varepsilon) = H_a^k(G^\varepsilon, \{0\})$.

Using part 1 of Lemma A.3, we represent $f'_{i\mu} \in W^{2m-m_{i\mu}-1/2}(\Gamma^\varepsilon)$ near the origin as follows:

$$f'_{i\mu}(r) = P_1(r) + f''_{i\mu}(r), \quad 0 < r < \varepsilon,$$

where $P_1(r)$ is a polynomial of order $2m - m_{i\mu} - 2$, whereas $f''_{i\mu} \in H_{b+1}^{2m-m_{i\mu}-1/2}(\Gamma^\varepsilon)$ (in fact, we can replace $b+1$ by any positive number in the last relation). Now we have $f'_{i\mu}, f''_{i\mu} \in H_{b-m+1}^{m-m_{i\mu}-1/2}(\Gamma^\varepsilon)$; therefore, $P_1 \in H_{b-m+1}^{m-m_{i\mu}-1/2}(\Gamma^\varepsilon)$, i.e., P_1 consists of monomials of order greater than or equal to $m - m_{i\mu} - 1$. This implies that $P_1 \in H_{b+1}^{2m-m_{i\mu}-1/2}(\Gamma^\varepsilon)$. Using part 3 of Lemma A.3, we obtain

$$\|f'_{i\mu}\|_{H_{b+1}^{2m-m_{i\mu}-1/2}(\Gamma^\varepsilon)} \leq \|P_1\|_{H_{b+1}^{2m-m_{i\mu}-1/2}(\Gamma^\varepsilon)} + \|f''_{i\mu}\|_{H_{b+1}^{2m-m_{i\mu}-1/2}(\Gamma^\varepsilon)} \leq k_6 \|f'_{i\mu}\|_{W^{2m-m_{i\mu}-1/2}(\Gamma^\varepsilon)}.$$

Combining this estimate with (2.18) yields (2.19).

3. Applying Lemma 1.2 to problem (2.16) and taking into account (2.15), (2.17), (2.19), and (2.14), we obtain

$$\|v\|_{H_{b+1}^{2m}(G)} \leq k_7 \left(\|f'\|_{H_{b+1}^0(G)} + \sum_{i,\mu} \|f'_{i\mu}\|_{H_{b+1}^{2m-m_{i\mu}-1/2}(\Gamma_i)} + \|v\|_{H_{b-2m+1}^0(G)} \right) \leq k_8 (\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}).$$

Combining this inequality with (2.14) yields (2.12). \square

The following corollary results from Lemma 2.3.

Corollary 2.1. *Let $A'(y, D_y)$ be the differential operator of order $2m - 1$, given by (2.1). Then*

$$\|A'(y, D_y)u\|_{H_{b+1}^1(G)} \leq c (\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}) \quad \forall u \in D(\mathbf{Q}). \quad (2.20)$$

Now we can prove that lower-order perturbations in (1.4) do not change the index of \mathbf{Q} .

Lemma 2.4. *Let the line $\operatorname{Im} \lambda = a + 1 - 2m$ contain no eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. Then the operators \mathbf{Q} and \mathbf{Q}' have the Fredholm property and $\operatorname{ind} \mathbf{Q}' = \operatorname{ind} \mathbf{Q}$.*

Proof. By Lemma 2.1, \mathbf{Q} and \mathbf{Q}' have the Fredholm property.

Introduce the operator $\mathbf{A}' : D(\mathbf{A}') \subset L_2(G) \rightarrow H_a^0(G)$ given by $\mathbf{A}'u = \mathbf{A}'(y, D_y)u$, $u \in D(\mathbf{A}') = D(\mathbf{Q})$. It follows from Corollary 2.1 and from the compactness of the embedding $H_{b+1}^1(G) \subset H_a^0(G)$ (see (2.9) and Lemma A.1) that $\mathbf{Q}' = \mathbf{Q} + \mathbf{A}'$ and \mathbf{A}' is a \mathbf{Q} -compact operator. Therefore, by Theorem A.4, we have $\operatorname{ind} \mathbf{Q}' = \operatorname{ind} \mathbf{Q}$. \square

Proof of Theorem 2.1. It follows from Lemma 2.1 in [17] that the spectrum of $\tilde{\mathcal{L}}(\lambda)$ is discrete. Therefore, one can find a number a such that $m - 1 < a < m$ and the line $\operatorname{Im} \lambda = a + 1 - 2m$ contains no eigenvalues of $\tilde{\mathcal{L}}(\lambda)$. In this case, Lemmas 2.1 and 2.4 imply $\operatorname{ind} \mathbf{P}' = \operatorname{ind} \mathbf{Q}' = \operatorname{ind} \mathbf{Q} = \operatorname{ind} \mathbf{P}$. \square

3 Perturbations in Nonlocal Conditions

3.1 Formulation of the main result

In this section, we investigate the stability of index for nonlocal operators under the perturbation of nonlocal conditions by operators which have the same form as $\mathbf{B}_{i\mu}^1$ and $\mathbf{B}_{i\mu}^2$. This situation is more difficult than that in Sec. 2 because the above perturbations explicitly change the domain of the corresponding unbounded operators. Therefore, these perturbations cannot be treated as relatively compact ones, and we make use of another approach based on the notion of the *gap between closed operators*.

We consider differential operators $C_{i\mu s}(y, D_y)$, $i = 1, \dots, N$, $\mu = 1, \dots, m$, $s = 1, \dots, S'_i$, of the same order $m_{i\mu}$ as $B_{i\mu s}$ in Sec. 1.1, given by

$$C_{i\mu s}(y, D_y)u = \sum_{|\alpha| \leq m_{i\mu}} c_{i\mu s\alpha}(y) D^\alpha u,$$

where $c_{i\mu s\alpha} \in C^\infty(\mathbb{R}^2)$. Introduce the operator $\mathbf{C}_{i\mu}^1$ by the formula

$$\mathbf{C}_{i\mu}^1 u = \sum_{s=1}^{S'_i} (C_{i\mu s}(y, D_y)(\zeta u))(\Omega'_{is}(y)), \quad y \in \Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K}), \quad \mathbf{C}_{i\mu}^1 u = 0, \quad y \in \Gamma_i \setminus \mathcal{O}_\varepsilon(\mathcal{K}),$$

where ζ and ε are the same as in the definition of $\mathbf{B}_{i\mu}^1$, whereas Ω'_{is} are C^∞ -diffeomorphisms possessing the same properties as Ω_{is} (in particular, they satisfy Condition 1.2 with S_i and Ω_{is} replaced by S'_i and Ω'_{is}).

We also consider operators $\mathbf{C}_{i\mu}^2$ satisfying Condition 1.3 with $\mathbf{B}_{i\mu}^2$ replaced by $\mathbf{C}_{i\mu}^2$. Set

$$\mathbf{C}_{i\mu} = \mathbf{C}_{i\mu}^1 + \mathbf{C}_{i\mu}^2.$$

We prove an index stability theorem under the following conditions (which are assumed to hold along with Conditions 1.1–1.3 throughout this sections, including the formulation of lemmas).

Condition 3.1 (see, e.g., [13]). *The system $\{\mathbf{B}_{i\mu}^0\}_{\mu=1}^m$ is normal on $\overline{\Gamma_i}$, $i = 1, \dots, N$.*

Condition 3.2. $D^\sigma c_{i\mu s\alpha}(g_{i1}) = D^\sigma c_{i\mu s\alpha}(g_{i2}) = 0$, $|\sigma| = 0, \dots, (m - 1) - (m_{i\mu} - |\alpha|)$.

Denote by g_{i1} and g_{i2} the end points of $\overline{\Gamma_i}$. Let τ_{i1} (τ_{i2}) be a unit vector tangent to $\overline{\Gamma_i}$ at the point g_{i1} (g_{i2}).

Condition 3.3. $\left. \frac{\partial^\beta \mathbf{C}_{i\mu}^2 u}{\partial \tau_{i1}^\beta} \right|_{y=g_{i1}} = \left. \frac{\partial^\beta \mathbf{C}_{i\mu}^2 u}{\partial \tau_{i2}^\beta} \right|_{y=g_{i2}} = 0, \quad \beta = 0, \dots, m-1-m_{i\mu}, \forall u \in W^{2m}(G \setminus \overline{\mathcal{O}_{\mathcal{K}_1}(\mathcal{K})}).$

The following lemma is a consequence of Conditions 3.2 and 3.3 (recall that $m-1 < a < m$ throughout).

Lemma 3.1. *The following inequalities hold:*

$$\|\mathbf{C}_{i\mu}^1 u\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)} \leq c_1 \|u\|_{H_{a+m}^{2m}(G)}, \quad (3.1)$$

$$\|\mathbf{C}_{i\mu}^2 u\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)} \leq c_2 \|u\|_{W^{2m}(G \setminus \overline{\mathcal{O}_{\mathcal{K}_1}(\mathcal{K})})}. \quad (3.2)$$

Proof. 1. For any $u \in H_{a+m}^{2m}(G)$, we have $(D^\alpha u)(\Omega'_{is}(y))|_{\Gamma_i} \in H_{a+m}^{2m-|\alpha|-1/2}(\Gamma_i) \subset H_{a+m-(m_{i\mu}-|\alpha|)}^{2m-m_{i\mu}-1/2}(\Gamma_i)$. Therefore, by Condition 3.2 and Lemma A.5, we have $(c_{i\mu\alpha} D^\alpha u)(\Omega'_{is}(y))|_{\Gamma_i} \in H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)$. Estimate (3.1) follows from the boundedness of the above embedding and from inequality (A.5).

2. It follows from Condition 1.3 (applied to $\mathbf{C}_{i\mu}^2$) that $\mathbf{C}_{i\mu}^2 u \in W^{2m-m_{i\mu}-1/2}(\Gamma_i)$ for any $u \in W^{2m}(G \setminus \overline{\mathcal{O}_{\mathcal{K}_1}(\mathcal{K})})$. Now it follows from Condition 3.3 and from Lemma A.4 that $\mathbf{C}_{i\mu}^2 u \in H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)$. Estimate (3.2) follows from inequality (1.2) (applied to $\mathbf{C}_{i\mu}^2$) and from (A.2). \square

In this section, we write $\mathbf{A} = \mathbf{A}(y, D_y)$. Consider the operators $\mathbf{P}_t : D(\mathbf{P}_t) \subset L_2(G) \rightarrow L_2(G)$, $t \in \mathbb{C}$, given by

$$\mathbf{P}_t u = \mathbf{A}u, \quad u \in D(\mathbf{P}_t) = \{u \in W^m(G, \mathbf{B} + t\mathbf{C}) : \mathbf{A}u \in L_2(G)\},$$

where $W^m(G, \mathbf{B} + t\mathbf{C})$ is the space of functions $u \in W^m(G)$ that satisfy the nonlocal conditions $(\mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + t\mathbf{C}_{i\mu})u = 0$. The main result of this section (to be proved in Sec. 3.2) is as follows.

Theorem 3.1. *Let Conditions 1.1–1.3 and 3.1–3.3 hold. Then $\text{ind } \mathbf{P}_t = \text{const } \forall t \in \mathbb{C}$.*

3.2 The gap between nonlocal operators in weighted spaces

As in Sec. 2, we preliminarily study the operators $\mathbf{Q}_t : D(\mathbf{Q}_t) \subset L_2(G) \rightarrow H_a^0(G)$ given by

$$\mathbf{Q}_t u = \mathbf{A}u, \quad u \in D(\mathbf{Q}_t) = \{u \in W^m(G, \mathbf{B} + t\mathbf{C}) : \mathbf{A}u \in H_a^0(G)\},$$

where $t \in \mathbb{C}$ and $W^m(G, \mathbf{B} + t\mathbf{C})$ is the same as in the definition of the operator \mathbf{P}_t . The operators \mathbf{P}_t and \mathbf{Q}_t correspond to the problem

$$\mathbf{A}u = f(y) \quad (y \in G), \quad (3.3)$$

$$(\mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + t\mathbf{C}_{i\mu})u = 0 \quad (y \in \Gamma_i; \ i = 1, \dots, N; \ \mu = 1, \dots, m). \quad (3.4)$$

Remark 3.1. The operator $\tilde{\mathcal{L}}(\lambda)$ was constructed in Sec. 1.2 by means of principal homogeneous parts of the operators \mathbf{A} and $B_{i\mu s}(y, D_y)$ at the points of the set \mathcal{K} . Due to Condition 3.2, the principal homogeneous parts of the operators $C_{i\mu s}(y, D_y)$ are equal to zero. Therefore, one and the same operator $\tilde{\mathcal{L}}(\lambda)$ corresponds to problem (3.3), (3.4) for any t .

Fix a number a such that $m-1 < a < m$ and the line $\text{Im } \lambda = a+1-2m$ contains no eigenvalues of $\tilde{\mathcal{L}}(\lambda)$ (which is possible due to the discreteness of the spectrum of $\tilde{\mathcal{L}}(\lambda)$). It follows from Remark 3.1 and from Lemma 2.1 that \mathbf{Q}_t has the Fredholm property. Therefore, its graph $\text{Gr } \mathbf{Q}_t$ is a closed subspace in the Hilbert space $L_2(G) \times H_a^0(G)$; this space is endowed with the norm

$$\|(u, f)\| = \left(\|u\|_{L_2(G)}^2 + \|f\|_{H_a^0(G)}^2 \right)^{1/2} \quad \forall (u, f) \in L_2(G) \times H_a^0(G).$$

Denote

$$\delta(\mathbf{Q}_t, \mathbf{Q}_{t+s}) = \sup_{u \in \mathbf{D}(\mathbf{Q}_t): \|(u, \mathbf{Q}_t u)\|=1} \text{dist}((u, \mathbf{Q}_t u), \text{Gr } \mathbf{Q}_{t+s}). \quad (3.5)$$

By Definition A.3, the number $\hat{\delta}(\mathbf{Q}_t, \mathbf{Q}_{t+s}) = \max\{\delta(\mathbf{Q}_t, \mathbf{Q}_{t+s}), \delta(\mathbf{Q}_{t+s}, \mathbf{Q}_t)\}$ is the *gap between the operators \mathbf{Q}_t and \mathbf{Q}_{t+s}* .

The main tool which enables us to prove the index stability theorem is Theorem A.5 and the following result (to be proved later on).

Theorem 3.2. *Let Conditions 1.1–1.3 and 3.1–3.3 hold. Suppose that the lines $\text{Im } \lambda = a + 1 - 2m$ and $\text{Im } \lambda = a + 1 - m$ contain no eigenvalues of $\tilde{\mathcal{L}}(\lambda)$. Then*

$$\hat{\delta}(\mathbf{Q}_t, \mathbf{Q}_{t+s}) \leq c_t s, \quad |s| \leq s_t, \quad (3.6)$$

where $s_t > 0$ is sufficiently small, while $c_t > 0$ does not depend on s .

First, we prove several auxiliary results.

Lemma 3.2. *Let the line $\text{Im } \lambda = a + 1 - m$ contain no eigenvalues of $\tilde{\mathcal{L}}(\lambda)$. Then*

$$\|u\|_{H_{a+m}^{2m}(G)} \leq c_t \|(u, \mathbf{A}u)\| \quad \forall u \in \mathbf{D}(\mathbf{Q}_{t+s}), \quad (3.7)$$

where $c_t > 0$ does not depend on s and u , provided that $|s|$ is sufficiently small.

Proof. 1. Consider the bounded operator

$$\mathbf{M}_t = \{\mathbf{A}, \mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + t\mathbf{C}_{i\mu}\} : H_{a+m}^{2m}(G) \rightarrow \mathcal{H}_{a+m}^0(G, \Gamma). \quad (3.8)$$

Since the belonging $v \in H_{a+m}^{2m}(G)$ implies $(\mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + t\mathbf{C}_{i\mu}^1)v \in H_{a+m}^{2m-m_{i\mu}-1/2}(\Gamma_i)$ and $\mathbf{C}_{i\mu}^2 v \in W^{2m-m_{i\mu}-1/2}(\Gamma_i) \subset H_{a+m}^{2m-m_{i\mu}-1/2}(\Gamma_i)$ (the latter relations are due to Condition 1.3 and part 1 of Lemma A.3), it follows that the operator \mathbf{M}_t is well defined.

By Theorem 6.1 in [5] and by Remark 3.1, the operator \mathbf{M}_t has the Fredholm property for any $t \in \mathbb{C}$. Therefore, applying Theorem A.1 and noting that the embedding $H_{a+m}^{2m} \subset L_2(G)$ is compact for $a < m$ (see Lemma A.1), we obtain

$$\|u\|_{H_{a+m}^{2m}(G)} \leq k_1 \left(\|\mathbf{M}_t u\|_{\mathcal{H}_{a+m}^0(G, \Gamma)} + \|u\|_{L_2(G)} \right) \quad \forall u \in H_{a+m}^{2m}(G), \quad (3.9)$$

where $k_1 > 0$ may depend on t but does not depend on s and u .

2. Now take a function $u \in \mathbf{D}(\mathbf{Q}_{t+s})$. By Lemma 2.3, $u \in H_{a+m}^{2m}(G)$. Inequality (3.9), estimate (1.2) (for $\mathbf{C}_{i\mu}^2$), and the boundedness of the embedding $W^{2m-m_{i\mu}-1/2}(\Gamma_i) \subset H_{a+m}^{2m-m_{i\mu}-1/2}(\Gamma_i)$ (see part 1 of Lemma A.3) yield

$$\|u\|_{H_{a+m}^{2m}(G)} \leq k_1 \left(\|\mathbf{A}u\|_{H_{a+m}^0(G)} + \|u\|_{L_2(G)} \right) + k_2 |s| \cdot \|u\|_{H_{a+m}^{2m}(G)} \quad \forall u \in \mathbf{D}(\mathbf{Q}_{t+s}),$$

where $k_2 > 0$ may depend on t but does not depend on s and u . Choosing $|s| \leq 1/(2k_2)$ and noting that the embedding $H_a^0(G) \subset H_{a+m}^0(G)$ is bounded, we obtain (3.7). \square

Lemmas 3.1 and 3.2 imply:

Corollary 3.1. *Let the line $\text{Im } \lambda = a + 1 - m$ contain no eigenvalues of $\tilde{\mathcal{L}}(\lambda)$. Then*

$$\|\mathbf{C}_{i\mu} u\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)} \leq c_t \|(u, \mathbf{A}u)\| \quad \forall u \in \mathbf{D}(\mathbf{Q}_{t+s}), \quad (3.10)$$

where $c_t > 0$ does not depend on s and u , provided that $|s|$ is sufficiently small.

The following two lemmas enable us to reduce nonlocal problems with nonhomogeneous nonlocal conditions to nonlocal problems with homogeneous ones. This is the place where Condition 3.1 is needed.

Lemma 3.3 (see Lemma 8.1 in [5]). *Let $a \in \mathbb{R}$. For any right-hand sides $f_{j\sigma\mu} \in H_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})$ in (1.7) such that $\text{supp } f_{j\sigma\mu} \subset \gamma_{j\sigma}^{\varepsilon/2}$, there exist functions $U_j \in H_a^{2m}(K_j)$ such that $\text{supp } U_j \subset \overline{K_j^\varepsilon}$,*

$$B_{j\sigma\mu j0}(y, D_y)U_j(y) = f_{j\sigma\mu}(y), \quad (B_{j\sigma\mu ks}(y, D_y)U_k)(\mathcal{G}_{j\sigma ks}y) = 0, \quad y \in \gamma_{j\sigma}, \quad (k, s) \neq (j, 0),$$

$$\sum_j \|U_j\|_{H_a^{2m}(K_j)} \leq c \sum_{j,\sigma,\mu} \|f_{j\sigma\mu}\|_{H_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})}.$$

Lemma 3.4. *Let $f_{i\mu} \in H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)$. Then, for $t \in \mathbb{C}$ and $|s| \leq 1$, there is a function $u \in H_a^{2m}(G)$ such that*

$$(\mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + (t+s)\mathbf{C}_{i\mu})u = f_{i\mu}, \quad (3.11)$$

$$\|u\|_{H_a^{2m}(G)} \leq c_t \sum_{i,\mu} \|f_{i\mu}\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)}, \quad (3.12)$$

where $c_t > 0$ does not depend on $f_{i\mu}$ and s .

Proof. Using Lemma 3.3 and a partition of unity, we construct a function $v \in H_a^{2m}(G)$ such that

$$\text{supp } v \subset \overline{G} \setminus \overline{G_\rho}, \quad (3.13)$$

$$\mathbf{B}_{i\mu}^0 v = f_{i\mu}, \quad \mathbf{B}_{i\mu}^1 v = 0, \quad \mathbf{C}_{i\mu}^1 v = 0, \quad (3.14)$$

$$\|v\|_{H_a^{2m}(G)} \leq k_1 \sum_{i,\mu} \|f_{i\mu}\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)}, \quad (3.15)$$

where $k_1 > 0$ does not depend on $f_{i\mu}$, t , and s .

By (3.13) and (1.3), we have $\text{supp } \mathbf{C}_{i\mu}^2 v \subset \mathcal{O}_{\varkappa_2}(\mathcal{K})$. Moreover, by Lemma 3.1, $\mathbf{C}_{i\mu}^2 v \in H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)$. Therefore, using Lemma 3.3 and a partition of unity again, we construct a function $w \in H_a^{2m}(G)$ such that

$$\text{supp } w \subset \mathcal{O}_{\varkappa_1}(\mathcal{K}), \quad (3.16)$$

$$\mathbf{B}_{i\mu}^0 w = -(t+s)\mathbf{C}_{i\mu}^2 v, \quad \mathbf{B}_{i\mu}^1 w = 0, \quad \mathbf{C}_{i\mu}^1 w = 0, \quad (3.17)$$

$$\|w\|_{H_a^{2m}(G)} \leq k_1 \sum_{i,\mu} \|(t+s)\mathbf{C}_{i\mu}^2 v\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)}.$$

Using the relation $|s| \leq 1$ and inequalities (3.2) and (3.15), we infer from the last inequality

$$\|w\|_{H_a^{2m}(G)} \leq k_1 \sum_{i,\mu} (|t|+1) \|\mathbf{C}_{i\mu}^2 v\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)} \leq k_2 \|v\|_{H_a^{2m}(G)} \leq k_2 k_1 \sum_{i,\mu} \|f_{i\mu}\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)}, \quad (3.18)$$

where $k_2 > 0$ may depend on t but does not depend on $f_{i\mu}$ and s .

By (3.16) and (3.2), we have $\mathbf{C}_{i\mu}^2 w = 0$. It follows from this relation, from (3.14), and from (3.17) that $u = v + w$ satisfies (3.11). Inequality (3.12) follows from inequalities (3.15) and (3.18). \square

Remark 3.2. One can easily see that, if $(\mathbf{C}_{i\mu}^2 v)(y) = 0$ in $\mathcal{O}_\varkappa(\mathcal{K})$ for some $\varkappa > 0$ and for any $v \in W^{2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$, then Lemma 3.4 is true for any $a \in \mathbb{R}$.

Proof of Theorem 3.2. 1. We have to prove inequality (3.6) for the quantity $\hat{\delta}(\mathbf{Q}_t, \mathbf{Q}_{t+s})$ replaced by $\delta(\mathbf{Q}_t, \mathbf{Q}_{t+s})$ and $\delta(\mathbf{Q}_{t+s}, \mathbf{Q}_t)$. Let us prove the inequality

$$\delta(\mathbf{Q}_t, \mathbf{Q}_{t+s}) \leq c_t |s|, \quad |s| \leq s_t. \quad (3.19)$$

(The proof of the corresponding inequality for $\delta(\mathbf{Q}_{t+s}, \mathbf{Q}_t)$ can be carried out in a similar way.)

Fix an arbitrary number t and take a function $u \in D(\mathbf{Q}_t)$. According to the definition (3.5), it suffices to find a function $v_s \in D(\mathbf{Q}_{t+s})$ (which depends on u) such that

$$\|u - v_s\|_{L_2(G)} + \|\mathbf{A}u - \mathbf{A}v_s\|_{H_a^0(G)} \leq k_1 |s| \cdot \|(u, \mathbf{A}u)\|, \quad (3.20)$$

where $|s|$ is sufficiently small and $k_1, k_2, \dots > 0$ may depend on t but do not depend on u and s .

Let us search $v_s \in D(\mathbf{Q}_{t+s})$ in the form

$$v_s = u + w_s, \quad (3.21)$$

where $w_s \in H_a^{2m}(G)$ is a solution of the problem

$$\mathbf{A}w_s = \sum_{j=1}^{J_s} \beta_j^s f_j^s, \quad (\mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + (t+s)\mathbf{C}_{i\mu})w_s = -s\mathbf{C}_{i\mu}u; \quad (3.22)$$

the numbers J_s and β_j^s as well as the functions $f_j^s \in H_a^0(G)$ will be defined later in such a way that the solution $w_s \in H_a^{2m}(G)$ exists.

2. To solve problem (3.22), we first note that $\mathbf{C}_{i\mu}u \in H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)$ due to Corollary 3.1. Hence, we can apply Lemma 3.4 and construct a function $W_s \in H_a^{2m}(G)$ such that

$$(\mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + (t+s)\mathbf{C}_{i\mu})W_s = -s\mathbf{C}_{i\mu}u, \quad (3.23)$$

$$\|W_s\|_{H_a^{2m}(G)} \leq k_2 |s| \sum_{i,\mu} \|\mathbf{C}_{i\mu}u\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)}. \quad (3.24)$$

Combining (3.24) with (3.10), we obtain

$$\|W_s\|_{H_a^{2m}(G)} \leq k_3 |s| \cdot \|(u, \mathbf{A}u)\|. \quad (3.25)$$

Clearly, problem (3.22) is equivalent to the following one:

$$\mathbf{A}Y_s = -\mathbf{A}W_s + \sum_{j=1}^{J_s} \beta_j^s f_j^s, \quad (\mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + (t+s)\mathbf{C}_{i\mu})Y_s = 0, \quad (3.26)$$

where

$$Y_s = w_s - W_s \in H_a^{2m}(G). \quad (3.27)$$

3. To solve problem (3.26), we consider the bounded operator

$$\mathbf{L}_t = \{\mathbf{A}, \mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + t\mathbf{C}_{i\mu}\} : H_a^{2m}(G) \rightarrow \mathcal{H}_a^0(G, \Gamma). \quad (3.28)$$

Note that $\mathbf{C}_{i\mu}^2 v \in H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)$ for any $v \in H_a^{2m}(G)$ due to Lemma 3.1; for this reason, we can write $\mathcal{H}_a^0(G, \Gamma)$ instead of $\mathcal{H}_a^0(G, \Gamma) \dot{+} \mathcal{R}_a^0(G, \Gamma)$ in the definition of the operator \mathbf{L}_t (cf. (2.3)). It follows from Theorem 6.1 in [5] and from Remark 3.1 that the operator \mathbf{L}_t has the Fredholm property for any $t \in \mathbb{C}$.

Expand the space $H_a^{2m}(G)$ into the orthogonal sum $H_a^{2m}(G) = \ker \mathbf{L}_t \oplus E_t$, where E_t is a closed subspace in $H_a^{2m}(G)$. Clearly, the operator

$$\mathbf{L}'_t = \{\mathbf{A}, \mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + t\mathbf{C}_{i\mu}\} : E_t \rightarrow \mathcal{H}_a^0(G, \Gamma) \quad (3.29)$$

has the Fredholm property and its kernel is trivial. In particular, this means that

$$\|u\|_{H_a^{2m}(G)} \leq k_4 \|\mathbf{L}'_t u\|_{\mathcal{H}_a^0(G, \Gamma)} \quad \forall u \in E_t. \quad (3.30)$$

Let $J = \text{codim } \mathcal{R}(\mathbf{L}'_t)$. It follows from Lemma 3.1 and from Theorem A.3 that the operator

$$\mathbf{L}'_{ts} = \{\mathbf{A}, \mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + (t+s)\mathbf{C}_{i\mu}\} : E_t \rightarrow \mathcal{H}_a^0(G, \Gamma)$$

also has the Fredholm property, its kernel is trivial and $\text{codim } \mathcal{R}(\mathbf{L}'_{ts}) = J$, provided that $|s| \leq s_t$, where $s_t > 0$ is sufficiently small. Moreover, using estimates (3.30), (3.1), and (3.2), we have, for all $u \in E_t$,

$$\|u\|_{H_a^{2m}(G)} \leq k_4 \left(\|\mathbf{L}'_{ts} u\|_{\mathcal{H}_a^0(G, \Gamma)} + s_t \sum_{i, \mu} \|\mathbf{C}_{i\mu} u\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)} \right) \leq k_5 \left(\|\mathbf{L}'_{ts} u\|_{\mathcal{H}_a^0(G, \Gamma)} + s_t \|u\|_{H_a^{2m}(G)} \right).$$

Taking $s_t \leq 1/(2k_5)$, we obtain

$$\|u\|_{H_a^{2m}(G)} \leq k_6 \|\mathbf{L}'_{ts} u\|_{\mathcal{H}_a^0(G, \Gamma)} \quad \forall u \in E_t. \quad (3.31)$$

Since \mathbf{L}'_{ts} has the Fredholm property, the set $\{f \in H_a^0(G) : (f, 0) \in \mathcal{R}(\mathbf{L}'_{ts})\}$ is closed and is of finite codimension J_s in $H_a^0(G)$. It is easy to see that $J_s \leq J$.

Let $f_1^s, \dots, f_{J_s}^s$ be an orthogonal normalized basis for the space

$$H_a^0(G) \ominus \{f \in H_a^0(G) : (f, 0) \in \mathcal{R}(\mathbf{L}'_{ts})\}.$$

Set $\beta_j^s = (\mathbf{A}W_s, f_j^s)_{H_a^0(G)}$. In this case, problem (3.26) admits a unique solution $Y_s \in E_t$, and, by virtue of (3.31) and (3.25), we have

$$\|Y_s\|_{H_a^{2m}(G)} \leq k_6 \left(\|\mathbf{A}W_s\|_{H_a^0(G)} + \sum_{j=1}^{J_s} |\beta_j^s| \right) \leq k_7 |s| \cdot \|(u, \mathbf{A}u)\| + k_6 J \max\{\beta_1^s, \dots, \beta_{J_s}^s\}. \quad (3.32)$$

Estimating $\beta_j^s = (\mathbf{A}W_s, f_j^s)_{H_a^0(G)}$ by Schwarz' inequality and using (3.25), we obtain

$$|\beta_j^s| \leq \|\mathbf{A}W_s\|_{H_a^0(G)} \leq k_8 |s| \cdot \|(u, \mathbf{A}u)\|.$$

Combining this inequality with (3.32) yields

$$\|Y_s\|_{H_a^{2m}(G)} \leq k_9 |s| \cdot \|(u, \mathbf{A}u)\|. \quad (3.33)$$

4. Taking into account equality (3.27), we deduce from estimates (3.25) and (3.33)

$$\|w_s\|_{L_2(G)} \leq k_{10} \|w_s\|_{H_a^{2m}(G)} \leq k_{11} |s| \cdot \|(u, \mathbf{A}u)\|, \quad (3.34)$$

$$\|\mathbf{A}w_s\|_{H_a^0(G)} \leq k_{12} \|w_s\|_{H_a^{2m}(G)} \leq k_{12} k_{11} |s| \cdot \|(u, \mathbf{A}u)\|, \quad (3.35)$$

where $w_s = Y_s + W_s$ is a solution of problem (3.22).

It follows from the boundedness of the embedding $H_a^{2m}(G) \subset W^m(G)$ that the function v_s defined by (3.21) belongs to $W^m(G)$, and $v_s \in \mathbf{D}(\mathbf{Q}_{t+s})$ due to the second relation in (3.22). The desired inequality (3.20) follows from (3.21), (3.34), and (3.35). \square

Proof of Theorem 3.1. It follows from Lemma 2.1 in [17] that the spectrum of $\tilde{\mathcal{L}}(\lambda)$ is discrete. Therefore, one can find a number a such that $m - 1 < a < m$ and the lines $\text{Im } \lambda = a + 1 - 2m$ and $\text{Im } \lambda = a + 1 - m$ contain no eigenvalues of $\tilde{\mathcal{L}}(\lambda)$. Fix two arbitrary numbers $t_1, t_2 \in \mathbb{C}$. By Lemma 2.1 and Remark 3.1, the operators \mathbf{Q}_t have the Fredholm property for all t in the interval $I_{t_1 t_2} \subset \mathbb{C}$ with the end points t_1, t_2 . Covering each point of the interval $I_{t_1 t_2}$ by a disk of sufficiently small radius, choosing a finite subcovering of $I_{t_1 t_2}$, and applying Theorems 3.2 and A.5, we see that $\text{ind } \mathbf{Q}_{t_1} = \text{ind } \mathbf{Q}_{t_2}$. It follows from this fact and from Lemma 2.1 that $\text{ind } \mathbf{P}_{t_1} = \text{ind } \mathbf{P}_{t_2}$. \square

Remark 3.3. Theorems 2.1 and 3.1 remain true in the case where the set \mathcal{K} consists of finitely many disjoint orbits. The proofs need evident modifications.

A Appendix

A.1 Some Properties of Sobolev and Weighted Spaces

Let G and Γ_i be the same as in Sec. 1.

Lemma A.1 (see Lemma 3.5 in [11]). *Let $k_2 > k_1$ and $k_2 - a_2 > k_1 - a_1$. Then the space $H_{a_2}^{k_2}(G)$ is compactly embedded into $H_{a_1}^{k_1}(G)$.*

Fix an arbitrary index i and set $\Gamma = \Gamma_i$. Let $g \in \bar{\Gamma} \setminus \Gamma$. We assume throughout this section, without loss of generality, that $g = 0$ and Γ coincides with the axis Oy_1 in a sufficiently small neighborhood $\mathcal{O}_\varepsilon(0)$ of the origin. In this section, we use the notation

$$G^\varepsilon = G \cap \mathcal{O}_\varepsilon(0), \quad \Gamma^\varepsilon = \Gamma \cap \mathcal{O}_\varepsilon(0),$$

in which case $H_a^k(G^\varepsilon) = H_a^k(G^\varepsilon, \{0\})$.

Lemma A.2. *If $u \in W^k(G^\varepsilon)$, $k \geq 1$, then the following assertions are true:*

1. $u(y) = P(y) + v(y)$ for $y \in G^\varepsilon$, where $P(y) = \sum_{|\alpha| \leq k-2} p_\alpha y^\alpha$, $v \in W^k(G^\varepsilon) \cap H_\delta^k(G^\varepsilon) \forall \delta > 0$ (if $k = 1$, we set $P(y) \equiv 0$); in particular, $u \in H_{k-1+\delta}^k(G^\varepsilon)$;
2. $D^\alpha u|_{y=0} = D^\alpha P|_{y=0}$ for $|\alpha| \leq k-2$;
3. $\sum_{|\alpha| \leq k-2} |p_\alpha| + \|v\|_{H_\delta^k(G^\varepsilon)} \leq c_\delta \|u\|_{W^k(G^\varepsilon)}$, where $c_\delta > 0$ does not depend on u .

Proof follows from Lemma 4.9 in [11] for $k = 1$ and from Lemma 4.11 in [11] for $k \geq 2$.

Lemma A.3. *If $\psi \in W^{k-1/2}(\Gamma^\varepsilon)$, $k \geq 1$, then the following assertions are true:*

1. $\psi(r) = P_1(r) + \varphi(r)$ for $0 < r < \varepsilon$, where $P_1(r) = \sum_{\beta=0}^{k-2} p_\beta r^\beta$, $\varphi \in W^{k-1/2}(\Gamma^\varepsilon) \cap H_\delta^{k-1/2}(\Gamma^\varepsilon) \forall \delta > 0$ (if $k = 1$, we set $P_1(r) \equiv 0$); in particular, $\psi \in H_{k-1+\delta}^{k-1/2}(\Gamma^\varepsilon)$;
2. $(d^\beta \psi / dr^\beta)|_{r=0} = (d^\beta P_1 / dr^\beta)|_{r=0}$ for $\beta = 0, \dots, k-2$;
3. $\sum_{\beta=0}^{k-2} |p_\beta| + \|\varphi\|_{H_\delta^{k-1/2}(\Gamma^\varepsilon)} \leq c_\delta \|\psi\|_{W^{k-1/2}(\Gamma^\varepsilon)}$, where $c_\delta > 0$ does not depend on ψ .

Proof. Consider a function $u \in W^k(G^\varepsilon)$ such that $u|_{\Gamma^\varepsilon} = \psi$ and $\|u\|_{W^k(G^\varepsilon)} \leq 2\|\psi\|_{W^{k-1/2}(\Gamma^\varepsilon)}$. Now it remains to apply Lemma A.2. \square

Lemma A.4. Let $\psi \in W^{k-1/2}(\Gamma)$, $k \geq 2$, and let

$$\left. \frac{d^s \psi}{dr^s} \right|_{y=0} = 0, \quad s = 0, \dots, l, \quad (\text{A.1})$$

for a fixed $l \leq k-2$. Then $\psi \in H_{k-2-l+\delta}^{k-1/2}(\Gamma) \forall \delta > 0$ and

$$\|\psi\|_{H_{k-2-l+\delta}^{k-1/2}(\Gamma)} \leq c_\delta \|\psi\|_{W^{k-1/2}(\Gamma)}, \quad (\text{A.2})$$

where $c_\delta > 0$ does not depend on ψ .

Proof. It follows from relations (A.1) and from Lemma A.3 (parts 1 and 2) that

$$\psi(r) = \sum_{\beta=l+1}^{k-2} p_\beta r^\beta + \varphi(r), \quad 0 < r < \varepsilon, \quad (\text{A.3})$$

where

$$\varphi \in H_\delta^{k-1/2}(\Gamma^\varepsilon) \subset H_{k-2-l+\delta}^{k-1/2}(\Gamma^\varepsilon), \quad \delta > 0. \quad (\text{A.4})$$

If $l = k-2$, then the sum in (A.3) is absent and the lemma follows from (A.4) and from part 3 of Lemma A.3.

If $l \leq k-3$, then the sum comprises the terms r^β for $\beta \geq l+1$. One can directly verify that $r^\beta \in H_{k-2-l+\delta}^{k-1/2}(\Gamma^\varepsilon)$ for the above β and $\forall \delta > 0$. Therefore, combining (A.3) with (A.4) and with part 3 of Lemma A.3, we complete the proof. \square

Lemma A.5. Let $\psi \in H_{a+l}^{k-1/2}(\Gamma)$, $l, k \in \mathbb{N}$, $a \in \mathbb{R}$, and let $b \in C^\infty(\bar{\Gamma})$ be a compactly supported function satisfying the relations $\left. \frac{\partial^s b}{\partial r^s} \right|_{r=0} = 0$, $s = 0, \dots, l-1$. Then

$$\|b\psi\|_{H_a^{k-1/2}(\Gamma)} \leq c \|\psi\|_{H_{a+l}^{k-1/2}(\Gamma)}. \quad (\text{A.5})$$

Proof. Clearly, it suffices to carry out the proof for compactly supported functions ψ and for Q and Γ replaced by $K = \{y \in \mathbb{R}^2 : 0 < \omega < \omega_0\}$ and $\gamma = \{y \in \mathbb{R}^2 : \omega = 0\}$, respectively.

Denote by $\hat{b} \in C^\infty(\mathbb{R})$ an extension of $b(y_1)$ to \mathbb{R} and introduce the function $B(y_1, y_2) = \hat{b}(y_1)$ for $(y_1, y_2) \in \mathbb{R}^2$. Clearly, we have

$$B \in C^\infty(\bar{K}), \quad D^\sigma B|_{y=0} = 0, \quad |\sigma| \leq l-1. \quad (\text{A.6})$$

Let $u \in H_{a+l}^k(K)$ be a compactly supported extension of ψ to the angle K such that

$$\|u\|_{H_{a+l}^k(K)} \leq c_1 \|\psi\|_{H_{a+l}^{k-1/2}(\gamma)}. \quad (\text{A.7})$$

It follows from Teylor's formula and from (A.6) that $|D^\sigma B| = O(r^{l-|\sigma|})$ for any σ ; therefore,

$$\begin{aligned} \|Bu\|_{H_a^k(K)}^2 &= \sum_{|\alpha| \leq k} \int_K r^{2(a+|\alpha|-k)} |D^\alpha(Bu)|^2 dy \leq c_2 \sum_{|\sigma|+|\zeta| \leq k} \int_K r^{2(a+|\sigma|+|\zeta|-k)} |D^\sigma B|^2 |D^\zeta u|^2 dy \\ &\leq c_3 \sum_{|\zeta| \leq k} \int_K r^{2(a+l+|\zeta|-k)} |D^\zeta u|^2 dy = c_3 \|u\|_{H_{a+l}^k(K)}^2 \end{aligned}$$

(remind that u is compactly supported). Combining this estimate with (A.7), we finally obtain

$$\|b\psi\|_{H_a^{k-1/2}(\gamma)} \leq \|Bu\|_{H_a^k(K)} \leq c_3^{1/2} \|u\|_{H_{a+l}^k(K)} \leq c_3^{1/2} c_1 \|\psi\|_{H_{a+l}^{k-1/2}(\gamma)}.$$

\square

A.2 Some Properties of Fredholm Operators

Let H_1 and H_2 be Hilbert spaces, and let $P : D(P) \subset H_1 \rightarrow H_2$ be a linear (in general, unbounded) operator.

Definition A.1. The operator P is said *to have the Fredholm property* if it is closed, its image is closed, and the dimension of its kernel $\ker P$ and the codimension of its image $\mathcal{R}(P)$ are finite. The number $\text{ind } P = \dim \ker P - \text{codim } \mathcal{R}(P)$ is called an *index* of the Fredholm operator P .

Theorem A.1 (see Theorem 7.1 in [12]). *Let H be a Hilbert space such that H_1 is compactly embedded into H , and let the operator P be closed. Then $\dim \ker P < \infty$ and $\mathcal{R}(P) = \overline{\mathcal{R}(P)}$ iff*

$$\|u\|_{H_1} \leq c(\|Pu\|_{H_2} + \|u\|_H) \quad \forall u \in D(P).$$

The proof of the following result is contained in part 2 of the proof of Lemma 2.5 in [7].

Theorem A.2. *Let $\dot{P} : D(\dot{P}) \subset H_1 \rightarrow H_2$ be a Fredholm operator such that P is an extension of \dot{P} , i.e., $\dot{P} \subset P$. Suppose that $\dim \ker P < \infty$, $\mathcal{R}(P) = \overline{\mathcal{R}(\dot{P})}$, and $\text{codim } \mathcal{R}(P) < \infty$. Then the operator P is closed (hence, it has the Fredholm property).*

Let $A : D(A) \subset H_1 \rightarrow H_2$ be a linear operator.

Theorem A.3 (see Sec. 16 in [12]). *Let the operator P have the Fredholm property, A be bounded, and $D(A) = H_1$. Then the operator $P + A$ has the Fredholm property, $\text{ind}(P + A) = \text{ind } P$, $\dim \ker(P + A) \leq \dim \ker P$, and $\text{codim } \mathcal{R}(P + A) \leq \text{codim } \mathcal{R}(P)$, provided that $\|A\|$ is sufficiently small.*

Definition A.2 (see, e.g., [12, 10]). The operator A is said to be *relatively compact with respect to P* or simply *P -compact* if $D(P) \subset D(A)$ and, for any sequence $u_n \in D(P)$ with both $\{u_n\}$ and $\{Pu_n\}$ bounded, $\{Au_n\}$ contains a convergent subsequence.

Theorem A.4 (see Theorem 5.26 in Chap. 4 of [10]). *Suppose that the operator P has the Fredholm property and the operator A is P -compact. Then the operator $P + A$ also has the Fredholm property and $\text{ind}(P + A) = \text{ind } P$.*

Finally, we introduce a concept of a gap between closed operators. Let $S : D(S) \subset H_1 \rightarrow H_2$ be a linear operator. In the space $H_1 \times H_2$, we introduce the norm

$$\|(u, f)\| = (\|u\|_{H_1}^2 + \|f\|_{H_2}^2)^{1/2} \quad \forall (u, f) \in H_1 \times H_2.$$

Set $\delta(P, S) = \sup_{u \in D(P): \|(u, Pu)\|=1} \text{dist}((u, Pu), \text{Gr } S)$, where $\text{Gr } S$ is the graph of the operator S .

Definition A.3. The number $\hat{\delta}(P, S) = \max\{\delta(P, S), \delta(S, P)\}$ is called a *gap between the operators P and S* .

Theorem A.5 (see Theorem 5.17 in Chap. 4 of [10]). *Let the operator P have the Fredholm property and S be closed. Then the operator S has the Fredholm property, $\text{ind } S = \text{ind } P$, $\dim \ker S \leq \dim \ker P$, and $\text{codim } \mathcal{R}(S) \leq \text{codim } \mathcal{R}(P)$, provided that the gap $\hat{\delta}(P, S)$ is sufficiently small.*

The author is grateful to A. L. Skubachevskii for attention.

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